# Statistical description of eigenfunctions in chaotic and weakly disordered systems beyond universality

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We present a semiclassical approach to eigenfunction statistics in chaotic and weakly disordered quantum systems which goes beyond Random Matrix Theory, supersymmetry techniques, and existing semiclassical methods. The approach is based on a generalization of Berry's Random Wave Model, combined with a consistent semiclassical representation of spatial two-point correlations. We derive closed expressions for arbitrary wavefunction averages in terms of universal coefficients and sums over classical paths, which contain, besides the supersymmetry results, novel oscillatory contributions. Their physical relevance is demonstrated in the context of Coulomb blockade physics.

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Since Chladni's famous experiments on vibrating plates two centuries ago, the morphology of eigenfunctions in wave (mechanical) systems has aroused curiosity [1]. In particular, the statistical description of eigenfunctions in quantum (or wave) systems with diffusive or chaotic classical (or ray) dynamics has been an intensive research field for more than 20 years [2]. Besides the pure academic interest, the study of the spatial properties of irregular eigenfunctions provides specific theoretical input for experimentally relevant quantities in a wide range of disciplines [1] including optics, acoustics, oceanography [3], quantum chaos, and mesoscopic condensed matter [4]. In the latter, for instance, the energy levels of interacting electrons in quantum dots are influenced by wave function correlations entering into interaction matrix elements [5, 6, 7].

In the semiclassical limit [8], characterized by typical classical actions $S_{\rm cl}$ much larger than Planck's constant, i.e. $\hbar_{\rm eff} \equiv \hbar/S_{\rm cl} \ll 1$, statistical measures for the spatial structure of eigenfunctions can be deduced from purely classical quantities, though in a highly non-trivial way. The most spectacular example of such a connection is embodied in Berry's Random Wave Model (RWM) [9]: eigenfunctions of classically chaotic systems possess the same statistical properties as Gaussian random fields with a universal spatial two-point correlation function.

A theory beyond the universal results of the RWM has been rigorously derived for disordered systems by an exact mapping of the quantum problem onto a supersymmetric field theory, the Nonlinear Sigma Model [6]. Semiclassical in spirit, this Diffusive Sigma Model expresses all results in terms of the classical diffusion propagator, and its success has motivated efforts to extend such methods to clean chaotic systems. The results of such (still conjectured) Ballistic Sigma Models (B$\sigma$M) [6, 10] are obtained by replacing the diffusion propagator for the disordered, metallic regime by a suitable ballistic counterpart. Two features of the B$\sigma$M render calculations difficult, already at the level of the first non-universal contribution beyond the RWM [11]: (i) the statistical distribution of eigenfunctions is *not Gaussian*, in clear contrast to the RWM, and (ii) different eigenfunctions are *not independent*.

In this Letter we present an alternative approach to eigenfunction statistics in which different eigenfunctions are described as *independent Gaussian* fields [12]. Employing simple Gaussian integrations, we provide closed expressions for general wave function averages for chaotic systems in the semiclassical regime, in terms of universal coefficients and sums over classical paths. We demonstrate that an arbitrary average contains, besides the universal (RWM) result, a system-dependent contribution (e.g. from confinement) composed of a smooth (diagonal) and an oscillatory part. We resolve the apparent contradiction to the supersymmetry techniques by showing how the results of the B$\sigma$M beyond universality are retained from our result by neglecting all oscillatory contributions. This finding for chaotic dynamics is used to extend our method to the metallic regime of disordered systems, while the physical relevance of the oscillatory terms found is illustrated for Coulomb Blockade conductance peak statistics.

*Defining statistical averages.* Consider a set of $d$ normalized, real solutions $\psi_{n_\alpha}$ ($n_\alpha \in \{n_1, \ldots, n_d\}$) of the Schrödinger equation with non-degenerate, generally non-consecutive eigenvalues $E_{n_1} < \ldots < E_{n_d}$. We study expressions of the general form $F(\vec{\psi}_{n_1}, \ldots, \vec{\psi}_{n_d})$ where each entry is a vector $\vec{\psi}_{n_\alpha} = (\psi_{n_\alpha}(\vec{r}_1^{\,\alpha}), \ldots, \psi_{n_\alpha}(\vec{r}_{f_\alpha}^{\,\alpha}))$ with $f_\alpha$ components depending on different positions $\vec{r}_i^{\,\alpha}, i = 1, \ldots, f_\alpha$ (if $\vec{r}_i^{\,\alpha} = \vec{r}_j^{\,\beta}$ for $\alpha \neq \beta$ then $i = j$ by convention). Upon varying $n_\alpha$ inside each window $[\bar{n}_\alpha - N/2, \bar{n}_\alpha + N/2]$, while keeping all differences $n_\alpha - n_\beta$ fixed, the function $F(\vec{\psi}_{n_1}, \ldots, \vec{\psi}_{n_d})$ will exhibit fluctuations. The *spectral average* (indicated by caligraphic letters) of the function $F$ is then naturally defined as

$$\mathcal{F} = \frac{1}{N} \sum_{s=-N/2}^{N/2} F(\vec{\psi}_{\bar{n}_1+s}, \ldots, \vec{\psi}_{\bar{n}_d+s}) \,. \qquad (1)$$

It is essential to keep in mind that $\mathcal{F}$ is fluctuating itself,

depending on the size and location of the energy windows and on the set of positions $\vec{r}_i^{\,\alpha}$. In the semiclassical limit one has $\bar{n}_\alpha \gg N \gg 1$ for all $\alpha$.

Averages playing a key role are the *spatial two-point correlation matrices* $\mathbf{R}_\alpha$ with entries defined as

$$R_\alpha^{i,j} = \frac{1}{N} \sum_{s=-N/2}^{N/2} \psi_{\bar{n}_\alpha+s}(\vec{r}_i^{\,\alpha}) \psi_{\bar{n}_\alpha+s}(\vec{r}_j^{\,\alpha}) \,. \qquad (2)$$

*Local Gaussian conjecture.* A local Gaussian theory for the spectral averages is based on two assumptions: (i) $\vec{\psi}_{\bar{n}_\alpha-N/2}, \ldots, \vec{\psi}_{\bar{n}_\alpha+N/2}$ are $N$ realizations of an $f_\alpha$-dimensional random vector, denoted by $\vec{v}_\alpha$, with distribution $P_\alpha(\vec{v}_\alpha) = (2\pi)^{-f_\alpha/2}\sqrt{\det \mathbf{R}_\alpha^{-1}} \exp\left[-\frac{1}{2}\vec{v}_\alpha.\mathbf{R}_\alpha^{-1}\vec{v}_\alpha\right]$; (ii) $\vec{v}_\alpha$ and $\vec{v}_\beta$ are independent random vectors for $\alpha \neq \beta$. In the local Gaussian theory all averages can be expressed through the correlation matrices $\mathbf{R}_\alpha$ by means of

$$\mathcal{F}^G = \int F(\vec{v}_1, \ldots, \vec{v}_d) P_1(\vec{v}_1) \ldots P_d(\vec{v}_d)\, d\vec{v}_1 \ldots d\vec{v}_d \,. \quad (3)$$

The local Gaussian conjecture states that $\mathcal{F} = \mathcal{F}^G$, namely that *any two eigenfunctions of a classically chaotic system behave like two independent Gaussian random fields, each of them uniquely characterized by the exact two-point correlation functions.* This is a natural generalization of the ideas presented in [9, 14, 15, 16] with far reaching consequences when a subsequent semiclassical approximation is consistently used.

*Semiclassical expansion.* In the semiclassical regime each correlation matrix is expanded into a constant part, a leading-order (in $\hbar_{\rm eff}$) fluctuating part, and higher-order terms [17]: $\mathbf{R}_\alpha = A^{-1}\mathbf{I}_\alpha + A^{-1}\tilde{\mathbf{R}}_\alpha + O(\hbar_{\rm eff}^{3/2})$, where $\mathbf{I}_\alpha$ is a $f_\alpha \times f_\alpha$ unit matrix and $A$ the system area for the 2d case considered. The matrix $A^{-1}\mathbf{I}_\alpha$ defines a Gaussian distribution $P^{RMT}(\vec{v}_\alpha) = (A/2\pi)^{f_\alpha/2} \exp\left[-(A/2)\vec{v}_\alpha \cdot \vec{v}_\alpha\right]$ yielding the Random Matrix Theory (RMT) results $\mathcal{F}^{RMT}$ for the $\alpha$-th state. It is straight forward to factorize each probability distribution as

$$P_\alpha(\vec{v}_\alpha) = P^{RMT}(\vec{v}_\alpha) \frac{\exp\left[\frac{A}{2}\vec{v}_\alpha \cdot (\mathbf{I}_\alpha + \tilde{\mathbf{R}}_\alpha)^{-1} \tilde{\mathbf{R}}_\alpha \vec{v}_\alpha\right]}{\sqrt{\det(\mathbf{I}_\alpha + \tilde{\mathbf{R}}_\alpha)}} . \quad (4)$$

As a key step we note that the semiclassical approach consistently keeps terms up to second order in the fluctuating part of the correlation matrices, since the semiclassical approximation neglects terms of order $O(\hbar_{\rm eff}^{3/2})$ [8] while $\tilde{\mathbf{R}}_\alpha \sim O(\hbar_{\rm eff}^{1/2})$. Taylor expansion of the probability distributions in Eq. (4) to second order in $\tilde{\mathbf{R}}_\alpha$ and substitution into Eq. (3) yields

$$\begin{aligned} \mathcal{F}^G &= \mathcal{F}^{RMT} + \sum_{\alpha=1}^{d} \sum_{i,j=1}^{f_\alpha} \tilde{R}_\alpha^{i,j} \times \qquad (5) \\ &\times \left[ \mathcal{F}_{i,j}^{\alpha} + \sum_{k,l=1}^{f_\alpha} \mathcal{F}_{i,j,k,l}^{\alpha} \tilde{R}_\alpha^{k,l} + \sum_{\beta<\alpha}^{d} \sum_{k,l=1}^{f_\beta} \mathcal{F}_{i,j,k,l}^{\alpha,\beta} \tilde{R}_\beta^{k,l} \right] \end{aligned}$$

with universal (system-independent) coefficients

$$\begin{aligned} \mathcal{F}^{RMT} &= \langle F(\vec{v}_1, \ldots, \vec{v}_d) \rangle^{RMT} , \\ \mathcal{F}_{i,j}^{\alpha} &= \langle F(\vec{v}_1, \ldots, \vec{v}_d) q_{i,j}(\vec{v}_\alpha) \rangle^{RMT} , \\ \mathcal{F}_{i,j,k,l}^{\alpha} &= \langle F(\vec{v}_1, \ldots, \vec{v}_d) q_{i,j,k,l}(\vec{v}_\alpha) \rangle^{RMT} , \qquad (6) \\ \mathcal{F}_{i,j,k,l}^{\alpha,\beta} &= \langle F(\vec{v}_1, \ldots, \vec{v}_d) q_{i,j}(\vec{v}_\alpha) q_{k,l}(\vec{v}_\beta) \rangle^{RMT} . \end{aligned}$$

Here $\langle \ldots \rangle^{RMT}$ denotes an average with respect to the distribution $\prod_{\eta=1}^{d} P^{RMT}(\vec{v}_\eta)$, and $q(\vec{v})$ are simple polynomials of the components of its argument $\vec{v}$:

$$\begin{aligned} q_{i,j}(\vec{v}) &= \frac{1}{2}(A v_i v_j - \delta_{ij}) \,, \\ q_{i,j,k,l}(\vec{v}) &= \frac{1}{2}[q_{i,j}(\vec{v}) q_{k,l}(\vec{v}) + 2 q_{i,k}(\vec{v}) q_{j,l}(\vec{v})] \qquad (7) \\ &\quad - \frac{1}{4} A^2 v_i v_j v_k v_l \,. \end{aligned}$$

*Semiclassical correlation function.* The fluctuating part of the correlation matrix is obtained from the semiclassical Green function and is given by [14, 15]

$$\begin{aligned} \tilde{R}_\alpha^{i,j} &= -\delta_{i,j} + J_0\left(k_{n_\alpha} |\vec{r}_i^{\,\alpha} - \vec{r}_j^{\,\alpha}|\right) + \qquad (8) \\ &+ \left(\frac{2\hbar}{m^2\pi}\right)^{1/2} \sum_{\gamma_{i,j}} \Gamma\left(\frac{T_{\gamma_{i,j}}}{\tau_N}\right) \left|D_{\gamma_{i,j}}\right|^{1/2} \cos\left(\frac{S_{\gamma_{i,j}}}{\hbar}\right) \end{aligned}$$

as a sum over classical trajectories $\gamma_{i,j}$ joining $\vec{r}_i^{\,\alpha}$ with $\vec{r}_j^{\,\alpha}$ at fixed energy $E_{n_\alpha} \simeq \bar{n}_\alpha \Delta$, where $\Delta$ denotes the (constant) mean level spacing. The Bessel function $J_0(x)$ with $k_{n_\alpha} = \sqrt{2mE_{n_\alpha}}/\hbar$ is the contribution from the unique direct trajectory between $\vec{r}_i^{\,\alpha}$ and $\vec{r}_j^{\,\alpha}$. The sum in Eq. (8) is taken over all non-direct orbits $\gamma_{i,j}$ with actions $S_{\gamma_{i,j}}(\vec{r}_i^{\,\alpha}, \vec{r}_j^{\,\alpha}, E_{n_\alpha}) = \int_{\gamma_{i,j}} \vec{p} \cdot d\vec{r}$ and semiclassical prefactors $D_{\gamma_{i,j}}$ where the stability and topology of each path enters [8]. The window function $\Gamma(x) = \sin(x)/x$ suppresses contributions from trajectories with traversal time $T_{\gamma_{i,j}} = \partial S_{\gamma_{i,j}}/\partial E_{n_\alpha}$ larger than the characteristic time $\tau_N = 2\hbar/N\Delta$ related to the energy average (1).

Equation (5), supplemented by the definitions (6,7) and Eq. (8) for $\tilde{R}_\alpha^{i,j}$, is our main result. We illustrate its power by computing the intensity distribution $I(\psi(\vec{r}_1)) = \delta(t - A\psi(\vec{r}_1)^2)$, a prominent and frequently studied [2, 6] measure for wave function statistics. While the evaluation based on supersymmetry methods is quite involved [11], using our Gaussian approach (with $d = 1, f_1 = 1$) we trivially find for the universal polynomials (7) $q_{1,1}(v) = \frac{1}{2}(Av^2 - 1)$ and $q_{1,1,1,1}(v) = \frac{3}{8}(Av^2 - 1)^2 - \frac{1}{4}A^2v^4$. The coefficients (6) are then obtained without any further integration, and Eq. (5) yields the intensity distribution

$$\mathcal{I}^G(t) = \frac{e^{-\frac{t}{2}}}{\sqrt{2\pi t}} \left[1 + \frac{\tilde{R}^{1,1}}{2}(t-1) + \frac{(\tilde{R}^{1,1})^2}{4}(3 - 6t + t^2)\right] \qquad (9)$$

in terms of closed orbits (through $\tilde{R}^{1,1}$) starting and ending at $\vec{r}_1$. Equation (9) includes both the universal limit

$e^{-\frac{t}{2}}/\sqrt{2\pi t}$ (the Porter-Thomas distribution [4]) and the sigma model result of [11] as we will show below.

*Diagonal approximation and sigma model.* For energy windows of size $N_{\rm Th} > \sqrt{A}k_{n_\alpha}$, corresponding to $\tau_N$ in Eq.. (8) smaller than the time of flight through the system, i.e. the ballistic Thouless time $\tau_{\rm Th}$, all contributions beyond the direct path are damped out. In this universal regime the RWM predictions (consistent with RMT) are given by substitution of the direct path contribution to $\tilde{R}_{i,j}$ (first line in Eq. (8)) into Eqs. (4) or (5). It has been shown that the Gaussian and sigma-model results coincide at this universal level [11, 18]. For smaller energy windows, however, system-dependent deviations appear, i.e., $\mathcal{F} = \mathcal{F}^{RWM} + \mathcal{F}^{SYS}$. In the present Gaussian approach deviations $\mathcal{F}^{G,SYS}$ from universality are obtained by substitution of the non-direct contribution to $\tilde{R}_{i,j}$ (second line of Eq. (8)) into Eq. (5). As we see, $\mathcal{F}^{G,SYS}$ consists of coherent single and double sums over non-direct classical paths of increasing length, while deviations $\mathcal{F}^{\sigma,SYS}$ from universality in the B$\sigma$M are expressed through a purely classical object, the ballistic propagator [11]. It is by no means clear whether the two approaches for $\mathcal{F}^{SYS}$ are consistent, and we address this fundamental issue now.

The frequency-dependent ballistic propagator $\Pi^{i,j}(w)$ is constructed by projecting the resolvent of the classical Liouville equation, $\{H(\vec{r},\vec{p}),\rho\} = iw\rho$ (where $\{\ldots\}$ is the Poisson bracket), onto configuration space at energy $E_{n_\alpha} = p_{n_\alpha}^2/2m$. The first step in unifying the local Gaussian and sigma-model approaches consists in expressing $\Pi^{i,j}(w)$ through a sum over non-zero paths [21], $\Pi^{i,j}(w) = \Pi_0^{i,j}(w) + \sum_{\gamma_{i,j}} D_{\gamma_{i,j}} e^{iwT_{\gamma_{i,j}}}$, where the contribution $\Pi_0$ from direct paths (set to zero when $\vec{r}_i = \vec{r}_j$) contains $D_{i,j}^0 = m^2/p_{n_\alpha}|\vec{r}_i - \vec{r}_j|$ and $T_{i,j}^0 = m|\vec{r}_i - \vec{r}_j|/p_{n_\alpha}$. The related energy-averaged, smoothed version reads

$$\tilde{\Pi}^{i,j}(w) = \Pi_0^{i,j}(w) + \sum_{\gamma_{i,j}} D_{\gamma_{i,j}} \Gamma^2 \left(\frac{T_{\gamma_{i,j}}}{\tau_N}\right) e^{iwT_{\gamma_{i,j}}} \quad (10)$$

with $\tilde{\Pi}^{i,j}(w) \simeq \Pi^{i,j}(w)$ for $\tau_N \gg \tau_{\rm Th}$. Consider now terms in the double sums in Eq. (5) where the classical paths involved join the same points, namely $\vec{r}_i^{\,\alpha} = \vec{r}_i^{\,\beta} = \vec{r}_i$ and $\vec{r}_j^{\,\alpha} = \vec{r}_j^{\,\beta} = \vec{r}_j$. The expression $\tilde{R}_\alpha^{i,j}\tilde{R}_\beta^{i,j}$ is a double sum over terms oscillatory in $[S_{\gamma_{i,j}}(E_{\bar{n}_\alpha}) \pm S_{\gamma_{i,j}}(E_{\bar{n}_\beta})]/\hbar$. When the energy difference $\hbar w_{\alpha,\beta} = \Delta(n_\beta - n_\alpha)$ is classically small, each trajectory in $\tilde{R}_\beta^{i,j}$ is a smooth deformation of a corresponding trajectory in $\tilde{R}_\alpha^{i,j}$, and we can expand $S_{\gamma_{i,j}}(E_{\bar{n}_\beta}) \simeq S_{\gamma_{i,j}}(E_{\bar{n}_\alpha}) + \hbar w_{\alpha,\beta} T_{\gamma_{i,j}}$. Collecting the non-oscillatory terms in the action difference of non-direct paths we obtain the so-called diagonal part

$$\left[\tilde{R}_\alpha^{i,j}\tilde{R}_\beta^{i,j}\right]_{\rm diag}^{SYS} = \frac{2\hbar}{m^2\pi}{\rm Re}\sum_{\gamma_{i,j}} D_{\gamma_{i,j}}\Gamma^2\left(\frac{T_{\gamma_{i,j}}}{\tau_N}\right)e^{iw_{\alpha,\beta}T_{\gamma_{i,j}}}. \quad (11)$$

Substitution of Eqs. (10,11) in Eq. (5) shows that the diagonal part of $\mathcal{F}^{G,SYS}$ is of order $\hbar_{\rm eff}$ and given by a linear combination of smoothed ballistic propagators (with universal coefficients $\mathcal{F}^{\alpha}_{i,j,k,l}, \mathcal{F}^{\alpha,\beta}_{i,j,k,l}$) with the direct path contribution excluded. Using the general formulas (6,7) the diagonal contribution to $\mathcal{F}^{G,SYS}$ is readily calculated, giving in the limit $\tau_N \gg \tau_{\rm Th}$ exactly the various specific B$\sigma$M results[19] available in the literature (moments of the wavefunction, distribution of intensities, two-energies four-point correlations [6], and two-point intensity distributions [20]). Hence we conclude that *to leading order in the deviation from universality, the B$\sigma$M corresponds to a Gaussian theory in diagonal approximation.* For instance, Eq. (9) yields $\hbar/(2\pi m^2)[\exp(-t/2)/\sqrt{2\pi t}](3-6t+t^2)$ for the prefactor of the ballistic propagator in the intensity distribution $\mathcal{I}^G(t)$, in perfect agreement with supersymmetry [11].

Here several remarks are due: (i) Possibly most importantly, our method provides also the general leading-order deviation from universality for the body of averages in disordered systems in the metallic regime by simply replacing $\tilde{\Pi}^{i,j}(w)$ by the diffusive propagator; (ii) there is not a single chaotic system where the exact ballistic propagator $\Pi^{i,j}(w)$ is known; hence the use of few classical paths to construct the smoothed version (10) makes the Gaussian theory more accessible for practical calculations; (iii) the direct path contribution, known to be counted twice as an artifact in the sigma model calculations [11], is correctly incorporated in the present approach; (iv) our results can be easily generalized to the case of broken time-reversal symmetry (by taking each $\vec{v}_\alpha$ in Eq. (3) as a complex vector with independent real and imaginary parts) and to the case of smooth potentials by using a sum over classical paths instead of Eq. (8).

*Statistics and oscillations of Coulomb Blockade transmission peaks.* As a further application of our approach we consider transport through a quantum dot weakly coupled to two leads. In this Coulomb Blockade regime, characterized by the mean resonance width $\bar{\Gamma} \ll k_{\rm B}T \ll \Delta$, transport is mediated by resonant tunneling with corresponding distinct conductance peaks. These Coulomb Blockade peaks and the fluctuations of their heights have been prominent objects of experimental studies [22]. The universal contribution to the conductance distribution, derived in RMT [23] for the case of one-channel leads, was extended to the multichannel case for ballistic quantum dots using the RWM [24] and for disordered systems using the sigma model [25], while effects due to periodic orbits where studied in [26]. However, for disordered dots in the Coulomb Blockade regime we are not aware of any prediction beyond universality. Moreover, non-universal oscillatory effects in ballistic dots were presented in [27], but the first-order theory used there fails to reproduce numerical results for the case of asymmetric leads [28].

Here we apply our approach to these two problems. To this end we consider leads supporting one channel each, connected to the dot at positions $\vec{r}_i$ ($i = 1, 2$) with equal coupling strength $\alpha$, generalizations to more

channels are straight forward [13]. Following [23, 24], the height of the $n_\alpha$-th conductance peak is given by $G_{n_\alpha} = (e^2/h)(\alpha\Delta)/(2\pi k_{\rm B}T) g_{n_\alpha}$ where

$$g_{n_\alpha} = \pi A \frac{\psi^2_{n_\alpha}(\vec{r}_1)\psi^2_{n_\alpha}(\vec{r}_2)}{\psi^2_{n_\alpha}(\vec{r}_1)+\psi^2_{n_\alpha}(\vec{r}_2)}, \qquad (12)$$

which fluctuates with $n_\alpha$. Applying our general Eq. (5) we obtain the distribution of conductances as $P(g) = (2\pi g)^{-1/2}\exp\left[-g/2\right]\left[1+\delta P^{(1)}(g)+\delta P^{(2)}(g)\right]$ with first and second order deviations from universality given by

$$\begin{aligned} \delta P^{(1)}(g) &= q^{(1)}(g)\left[\tilde{R}^{1,1}+\tilde{R}^{2,2}\right], \\ \delta P^{(2)}(g) &= q_1^2(g)\left[(\tilde{R}^{1,1})^2+(\tilde{R}^{2,2})^2\right]+ \\ &+ q_2^{(2)}(g)\tilde{R}^{1,1}\tilde{R}^{2,2}+q_3^{(2)}(g)(\tilde{R}^{1,2})^2\,. \end{aligned} \qquad (13)$$

Here the polynomial corrections to the RMT result, $q^{(1)}(x)(1/4)(x-1), q_1^{(2)}(x) = (1/32)(x^2-9x+6), q_2^{(2)}(x) = (1/16)(x^2-3x)$, and $q_3^{(2)} = (1/8)(x^2-7x+4)$, are modulated by the single and double sums over classical paths.

Using $P(g)$, we now calculate the mean of the conductance peak heights, $\langle g\rangle = 1+\delta g^{(1)}+\delta g^{(2)}$. First we discuss the leading term $\delta g^{(1)} = (1/2)(\tilde{R}^{1,1}+\tilde{R}^{2,2}) \sim O(\hbar_{\rm eff}^{1/2})$. In a billiard system with energy window $\delta e \simeq \hbar/\tau_{\rm Th}$ we have $\tilde{R}^{i,i} \sim \cos(kL_i)$ where $L_i$ is the length of the shortest classical trajectory starting and ending at $\vec{r}_i$. The corresponding modulations of the conductance with frequencies $L_1, L_2$ were already reported in [27, 28].

Our approach now enables us to go beyond the leading semiclassical order and to provide the explicit expression $\delta g^{(2)} = -(3/16)(\tilde{R}^{1,1}-\tilde{R}^{2,2})^2-(1/4)(\tilde{R}^{1,2})^2 \sim O(\hbar_{\rm eff})$. As we see, $\delta g^{(2)}$ will be a combination of terms $\cos[k(L_i \pm L_j)]$. We conclude that the second-order calculation is essential to understand the modulations with frequency $L_1 \pm L_2$ numerically observed for asymmetric leads in Ref. [28]. To our knowledge neither the first- nor the second-order oscillatory effect, present in Coulomb blockade physics, is accessible using the B$\sigma$M.

For diffusive dots in the metallic regime $\delta g^{(1)}$ is exponentially supressed and we just replace in the diagonal contribution to $\delta g^{(2)}$ the smoothed ballistic propagator $\tilde{\Pi}^{i,j}(w)$ by its diffusive counterpart $\Pi^{i,j}_{\rm dis}(w)$ to obtain

$$\delta g^{\rm dis} = -\frac{3\hbar}{8\pi m^2}[\Pi^{1,1}_{\rm dis}(0)+\Pi^{2,2}_{\rm dis}(0)+4\Pi^{1,2}_{\rm dis}(0)]\,. \qquad (14)$$

Since $\Pi^{i,j}_{\rm dis}(0) > 0$, we predict that the leading non-universal correction to the mean conductance peak height in weakly disordered quantum dots (with single-channel point contacts) is *always negative* and of order $\hbar_{\rm eff}$.

To summarize, we have used a local Gaussian theory for eigenfunction statistics to derive both smooth and oscillatory effects beyond (and including) the universal Random Wave Model. Smooth contributions are shown to give existing results of the Sigma Model, illuminating the connection between the two methods beyond universality. In view of that, we use our approach to present new results for the conductance of diffusive quantum dots in the Coulomb Blockade regime. For the ballistic case, new oscillatory effects neglected both by the Sigma Model and previous semiclassical approaches are shown to describe previously unexplained numerical results.

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